

Joint measurability and temporal steering

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Quintino *et. al.* (Phys. Rev. Lett. **113**, 160402 (2014)) and Uola *et. al.* (Phys. Rev. Lett. **113**, 160403 (2014)) have recently established an intrinsic relation between non-joint measurability and Einstein-Podolsky-Rosen steering. They showed that a set of measurements is incompatible (i.e., not jointly measurable) if and only if it can be used for the demonstration of steering. In this paper, we prove the temporal analog of this result viz., a set of measurements are incompatible if and only if it exhibits *temporal steering* in a single quantum system.

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1. Introduction

The purpose of measurements is to discern the properties of a system under investigation. In the classical scenario, all the physical observables are *jointly measurable* (or compatible). In contrast, in the quantum world, non-commuting observables are declared to be *incompatible* because it is not possible to assign well-defined values to these observables jointly. The notion of *compatibility* of measurements is captured entirely by *commutativity* of the observables if one restricts only to projective valued (PV) measurements. However, connecting compatibility of measurements with commutativity turns out to be limited in an extended framework, where the conventional idea of *sharp* PV measurements of self adjoint observables gets broadened to include generalized measurements of positive operator valued observables^{1–11}. Active research efforts are dedicated^{1,4–18} to investigate a clear operational criteria of joint measurability within the generalized framework of POVMs and to identify the significance of *incompatible* measurements in revealing puzzling quantum features like Bell non-locality¹⁹, contextuality²⁰ and steering^{21,22}. Particularly, Wolf *et. al.*⁸ have proved that the violation of the Clauser-Horne-Shimony-Holt (CHSH)²³ inequality in an entangled state can be witnessed if and only if *incompatible* measurements of any pairs of positive operator valued measures (POVMs) with binary outcomes are employed. It has been realized that a generalized non-contextuality inequality can get violated by a quantum state in two dimensional²⁴ Hilbert

space by employing a set of three dichotomic POVMs, which have pairwise joint measurability – but no triple-wise joint measurability^{11,14}. In yet another recent development, Quintino *et. al.*¹⁶ and Uola *et. al.*¹⁷ have established a general result that a set of non-jointly measurable POVMs is the one, which is useful for the task of *non-local steering*.

The concept of *non-local Einstein-Podolsky-Rosen (EPR) steering* was originally initiated by Schrödinger²¹ – as the ability to remotely prepare the state of a subsystem of an entangled state by performing local measurements on another subsystem. An experimentally testable steering criteria was developed by Reid²⁵ for continuous variable systems (Reid proposed an uncertainty relation involving conditional variances of position and momentum violation of which demonstrates EPR non-locality in an entangled state; this is the first steering inequality though it was not explicitly stated in Ref.²⁵). Wiseman *et. al.*²⁶ formalized the task of steering via falsification of local hidden state (LHS) models and showed that steering constitutes a different class of non-locality that lies between entanglement and Bell non-locality.

The quantum steering task is usually described in terms of an example: Alice supposedly prepares a composite quantum state and sends a subsystem to Bob. She tries to convince Bob that they share an entangled state, which would allow her (with the help of local measurements on her part of the system) to remotely affect (steer) Bob's quantum state. In order to verify if Alice's claim is true, Bob asks Alice to perform a local measurement of an observable X_k on her part of the state and announce her result x_k . From local quantum state tomography (via measurements at his end), Bob determines his set of states $\{\rho_{x_k|k}/\text{Tr}[\rho_{x_k|k}]\}$. If Bob's assemblage¹⁶ i.e., the set of un-normalized states $\{\rho_{x_k|k}\}$ (realized in each experimental run for different choices of Alice's observables X_k with statistical outcomes x_k) does not admit a LHS

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decomposition of the form²⁷

$$\rho_{x_k|k} = \sum_{\lambda} g(\lambda) p(x_k|k, \lambda) \rho_{\lambda}, \quad (1)$$

(where $0 \leq g(\lambda) \leq 1$; $\sum_{\lambda} g(\lambda) = 1$ and $0 \leq p(x_k|k, \lambda) \leq 1$; $\sum_{x_k} p(x_k|k, \lambda) = 1$) then Bob can convince himself that Alice is not cheating him and they indeed share an entangled state. On the other hand, if Bob's assemblage $\{\rho_{x_k|k}\}$ admits a LHS structure (1), he would be unable to witness violation of any of the steering inequalities²² and so, concludes that Alice is fooling him.

In order to demonstrate steerability, (i) Alice and Bob should necessarily share an entangled state (though the converse is not true²⁶) and (ii) Alice's measurements must be comprised of incompatible POVMs^{16,17}. Thus, in addition to entanglement being a resource for the steering task, incompatibility of measurements too plays a key role.

Apart from the developments towards probing non-locality, in 1985 Leggett and Garg²⁸ had proposed an inequality to test the concept of *macro-realism* in a single quantum system in terms of the correlations in the statistical outcomes of a dynamical observable at different times²⁹. Macro-realism rests on the assumptions: (i) physical properties of a macroscopic object exist independent of the act of observation (postulate on the existence of *reality* for all physical observables in the macroscopic world) and (ii) measurements are *non-invasive* i.e., the measurements of an observable at different instants of time do not influence its subsequent dynamical evolution. It has been experimentally demonstrated in a wide range of quantum systems^{30–35} that *temporal* correlations in the outcomes of an observable measured sequentially on a quantum system at different times do not fall under the tenet of *macro-realism* i.e., they violate Leggett-Garg inequality (also termed as temporal Bell inequality^{36,37}). Very recently, the temporal analog of steering in a *single* quantum system has been proposed^{38,39}. In this paper we show that the connection between spatial steerability and incompatible measurements of Ref.¹⁶ and¹⁷ can be extended to its temporal counterpart in a single quantum system also viz., non-joint measurability is necessary and sufficient for temporal steerability.

We organize the contents of the paper as follows. We begin by giving a concise description of joint measurability of POVMs in Section 2. Section 3 is devoted to discussing temporal analog of EPR steering in a single quantum system and to establish that a set of incompatible POVMs is necessary and sufficient for temporal steering. An example to bring forth the intrinsic connection between temporal steering and incompatibility of measurements in a single qubit system is discussed in the subsection 3.A. Concluding remarks are given in Section 4.

2. Joint measurability of POVMs

Mathematically, a POVM is a collection $\mathbb{E} = \{E(x)\}$ consisting of positive self-adjoint operators $E(x) \geq 0$ as its elements – which sum up to give the identity operator $\mathbb{1}$ i.e., $\sum_x E(x) = \mathbb{1}$. A measurement of \mathbb{E} in a quantum state ρ results in the outcomes x with probability of occurrence $p(x|E) = \text{Tr}[\rho E(x)]$. It may be noted that the POVM $\{E(x)\}$ encompasses the conventional PV measurements as a special case (when the elements $E(x)$ of the POVM constitute a complete and orthogonal set $\{\Pi(x)\}$ of projectors).

In the restricted measurement scenario, where only PV measurements are considered, it is well established that commuting physical observables are jointly measurable. But in the extended framework of generalized measurements, the notion of joint measurability is non-trivial and has received increasing attention^{1–11}. A more refined notion of compatible (jointly measurable) POVMs is defined as follows. A set of POVMs $\mathbb{E}_k = \{E_k(x_k)\}$ is said to be compatible if the probabilities $p(x_k|k) = \text{Tr}[\rho E_k(x_k)]$ of their outcomes in any arbitrary quantum state ρ can be discerned by measuring a *global* POVM $\mathbb{G} = \{G(\lambda); G(\lambda) \geq 0, \sum_{\lambda} G(\lambda) = \mathbb{1}\}$ – where the measurement outcomes $\lambda = \{x_1, x_2, \dots\}$ of $G(\lambda)$ occur with probabilities $g(\lambda) = \text{Tr}[\rho G(\lambda)]$ – by classical post processing of the data^{7,17}:

$$p(x_k|k) = \sum_{\lambda} g(\lambda) p(x_k|k, \lambda) \quad (2)$$

where $0 \leq p(x_k|k, \lambda) \leq 1$ in (2) are some arbitrary positive numbers satisfying $\sum_{x_k} p(x_k|k, \lambda) = 1$.

More precisely, associated with a set of jointly measurable POVMs $\{E_k(x_k)\}$ there exists a grand POVM $\mathbb{G} = \{G(\lambda)\}$ such that

$$E_k(x_k) = \sum_{\lambda} p(x_k|k, \lambda) G(\lambda) \quad \forall k. \quad (3)$$

In other words, it suffices to measure the grand POVM \mathbb{G} to discern the measurement results of compatible POVMs \mathbb{E}_k .

An important aspect to be highlighted here is that the generalized POVMs are jointly measurable even if they do not commute with each other.

Consider a triad of qubit observables $X = |0\rangle\langle 1| + |1\rangle\langle 0|$, $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ measured by employing the POVMs \mathbb{E}_X , \mathbb{E}_Y and \mathbb{E}_Z defined in terms of their elements

$$\begin{aligned} E_X(x) &= \frac{1}{2}(\mathbb{1} + \eta x X), \\ E_Y(y) &= \frac{1}{2}(\mathbb{1} + \eta y Y), \\ E_Z(z) &= \frac{1}{2}(\mathbb{1} + \eta z Z). \end{aligned} \quad (4)$$

The measurements result in binary outcomes $x, y, z = \pm 1$ and they correspond to *fuzzy* measurements of the

observables X, Y, Z , characterized by the *unsharpness* parameter $0 \leq \eta \leq 1$. It has been identified that the qubit POVMs $\{E_X(x)\}$, $\{E_Y(y)\}$ and $\{E_Z(z)\}$ are *pairwise* jointly measurable if and only if $\eta \leq 1/\sqrt{2}$ and the condition $\eta \leq 1/\sqrt{3}$ is necessary and sufficient for their triplewise joint measurability^{6,11,14}. It may also be noted that when $\eta = 1$, the POVMs $\{E_X(x)\}, \{E_Y(y)\}, \{E_Z(z)\}$ reduce to their corresponding *sharp* PV versions $\{\Pi_X(x)\}, \{\Pi_Y(y)\}, \{\Pi_Z(z)\}$.

3. Temporal steering and incompatible measurements

We consider a system prepared in a quantum state $\rho = \rho(0)$, which evolves under the Hamiltonian evolution $U(t) = e^{-iHt/\hbar}$, dynamically transforming the state (in the Schrodinger picture) as $\rho \rightarrow \rho(t) = U(t)\rho U^\dagger(t)$ at time t . The physical observables undergo dynamical evolution (in the Heisenberg picture) as $X(0) \rightarrow X(t) = U^\dagger(t)X(0)U(t)$. The observable X at different time instants t_k (which we denote by X_k) do not commute in general. Hence $\{X_k\}$ are not *jointly measurable* within the restricted framework of PV measurements. Contrast this situation with the classical scenario, where measurement of an observable at a given instant of time does not disturb its subsequent evolution. In other words, one can measure an observable at different instants of time jointly in the classical scenario. In the quantum case, measurements of non-commuting observables, in general, form an incompatible set of measurements.

To illustrate the temporal analog of steering, we consider a game involving two players Alice and Bob. Alice prepares a state ρ (which is not disclosed to Bob). Bob asks Alice to measure the observable X at different instants of time t_k using incompatible POVMs. Alice claims that she has measured X_k and obtained an outcome x_k with probability $p(x_k|k)$. She gives the post measured states $\frac{\rho_{x_k|k}}{\text{Tr}[\rho_{x_k|k}]}$ to Bob. Bob's task is to verify if Alice has given him a genuine set of post measured assemblage $\{\rho_{x_k|k}\}$, where the unnormalized states

$$\rho_{x_k|k} = \sqrt{E_k(x_k)}\rho\sqrt{E_k(x_k)} \quad (5)$$

have resulted via measurements of incompatible POVMs $\{E_k(x_k)\}$ of the observables X_k or if Alice is cheating him by merely stating that she has performed the measurements.

Bob can only trust his measurements on the states handed over to him by Alice, with a pre-label $\{x_k, p(x_k|k)\}$. In order to accomplish the protocol, Bob may choose to measure the observables X_l at a later time, $l \geq k$ on the assemblage $\rho_{x_k|k}$ and record the conditional probabilities $\mathcal{P}(x_l|x_k)$ of his outcomes x_l (given that Alice had obtained an outcome x_k in her measurement of the observable X_k); he then explores if the *temporal correlations* of the observables X_k, X_l violate any steering inequality⁴⁰. If the temporal steering inequality is violated, then Bob concludes that Alice has indeed

performed incompatible measurements of the observables X_k . We refer to this scenario as *temporal steering*⁴¹.

More generally, Bob could determine the assemblage $\{\rho_{x_k|k}\}$ given to him through quantum state tomography; if the assemblage $\rho_{x_k|k}$ is of the *hidden state* (HS) form, $\rho_{x_k|k} = \sum_\lambda g(\lambda)p(x_k|k, \lambda)\rho_\lambda$ (which is exactly identical to the LHS form (1)) where $0 \leq p(x_k|k, \lambda) \leq 1$, $\sum_{x_k} p(x_k|k, \lambda) = 1$, then Bob convinces himself that the assemblage $\{\rho_{x_k|k}\}$ is not *temporally steered*. Because the actual scenario may be the following. Alice could have drawn some random states ρ_λ with probability $g(\lambda)$ (from a statistical mixture $\rho = \sum_\lambda g(\lambda)\rho_\lambda$) – but announce that an outcome x_k has occurred in the measurement of the observable X at time t_k , with a probability of occurrence $p(x_k|k, \lambda)$ (Alice could have theoretically calculated the probabilities $p(x_k|k, \lambda)$ for the hypothetical outcomes x_k of measurement). If Alice has indeed performed incompatible measurements, as she claims, Bob's assemblage $\{\rho_{x_k|k}\}$ deviates from the HS form. Bob can then convince himself that Alice has indeed given him a set of states which reveals *temporal steering*, and it has resulted from the measurements of the observable X at different instants of time using incompatible POVMs.

We now proceed to show that measurements of $\{X_k\}$ using a *compatible* set $\{\mathbb{E}_k\}$ of POVMs do not lead to temporal steering.

Let us suppose that Alice performs measurement of a global POVM $\mathbb{G} = \{G(\lambda)\}$. After her measurement the post measured states are given by

$$\rho_\lambda = \sqrt{G(\lambda)}\rho\sqrt{G(\lambda)}/g(\lambda), \quad (6)$$

where $g(\lambda) = \text{Tr}[\rho G(\lambda)]$ is the probability of outcome λ . Alice would then classically post process the measurement data of the global POVM $\mathbb{G} = \{G(\lambda)\}$ to obtain the probabilities of outcomes $p(x_k|k)$ of measurement of any compatible POVMs \mathbb{E}_k to have resulted in an outcome x_k as,

$$\begin{aligned} p(x_k|k) &= \text{Tr}[\rho E_k(x_k)] \\ &= \sum_\lambda p(x_k|k, \lambda) \text{Tr}[\rho G(\lambda)] \\ &= \sum_\lambda p(x_k|k, \lambda) g(\lambda). \end{aligned} \quad (7)$$

More specifically, Alice could discern the results of measurements of *compatible* POVMs $\mathbb{E}_k = \{E_k(x_k)\}$ via measurement of a global POVM $\mathbb{G} = \{G(\lambda)\}$ and then using the decomposition $E_k(x_k) = \sum_\lambda p(x_k|k, \lambda) G(\lambda)$.

After Alice announces her measurement results $\{x_k, p(x_k|k)\}$ of $E_k(x_k)$ and hands over the post measured set of states, Bob detects that his assemblage $\{\rho_{x_k|k}\}$ is of the HS form $\rho_{x_k|k} = \sum_\lambda g(\lambda)p(x_k|k, \lambda)\rho_\lambda$. Thus, Bob concludes that there is no temporal steering.

Conversely, we prove that non-jointly measurable (incompatible) POVMs are sufficient to demonstrate temporal steering. To bring this out, we consider a completely

random state $\rho = \mathbb{1}/d$ and a set of POVMs $\{E_k\}$ for the measurements of the observables $\{X_k\}$. The post measured assemblage $\{\rho_{x_k|k}\}$ is characterized by its elements,

$$\begin{aligned}\rho_{x_k|k} &= \sqrt{E_k(x_k)} \rho \sqrt{E_k(x_k)} \\ &= \frac{1}{d} E_k(x_k).\end{aligned}\quad (8)$$

One can thus express the elements $E_k(x_k)$ of the POVM in terms of the assemblage $\{\rho_{x_k|k}\}$ as⁴³,

$$E_k(x_k) = d \rho_{x_k|k} \quad (9)$$

If there is no temporal steering, then the assemblage $\{\rho_{x_k|k}\}$ is described by a HS form (1) and hence one obtains

$$\begin{aligned}E_k(x_k) &= d \sum_{\lambda} g(\lambda) p(x_k|k, \lambda) \rho_{\lambda} \\ &= \sum_{\lambda} p(x_k|k, \lambda) G(\lambda)\end{aligned}\quad (10)$$

where $G(\lambda) = d g(\lambda) \rho_{\lambda}$. It is easy to see that (10) is essentially the joint measurability condition (see (3)) for the definition of compatible POVMs $\{E_k(x)\}$. We thus obtain the result: *A set of POVMs is said to be non-jointly measurable if and only if it is useful for demonstrating temporal steering.*

Our result highlights that one does not require a steerable entangled state to determine if a given set of measurements is compatible or not; it suffices to detect temporal non-steerability in a single quantum system itself to accomplish this task.

A. Joint measurability and temporal steering in a single qubit system

For the purpose of illustrating the intrinsic connection between temporal steering and non-joint measurability of the POVMs in a single qubit system, suppose that Alice prepares a single qubit system in a maximally disordered state $\rho = \frac{1}{2}$. Bob asks Alice to subject the system to a Hamiltonian evolution $U(t) = e^{-iHt/\hbar}$ where the Hamiltonian $H = \hbar\omega Z = \hbar\omega(|0\rangle\langle 0| - |1\rangle\langle 1|)$ and measure the observables $X_k = U^\dagger(t_k) X U(t_k) = X \cos(\omega t_k) + Y \sin(\omega t_k)$; $X = |0\rangle\langle 1| + |1\rangle\langle 0|$, $Y = -i(|0\rangle\langle 1| - |1\rangle\langle 0|)$ at two different time intervals (i) $t_1 = 0$ and (ii) $t_2 = \pi/(2\omega)$ using incompatible POVMs. Alice employs binary outcome POVMs $\{E_X(x) = \frac{1}{2}(\mathbb{1} + \eta x X), x = \pm 1\}$ and $\{E_Y(y) = \frac{1}{2}(\mathbb{1} + \eta y Y), y = \pm 1\}$ to measure the observables $X_1 = X$, $X_2 = Y$ respectively. After her measurements on several identically prepared copies of the initial state, Alice hands over four different assemblages $\{\rho_{x=\pm 1|1}\}$, $\{\rho_{y=\pm 1|2}\}$ – labelled by the outcomes of measurements $x = \pm 1$, $y = \pm 1$ and the corresponding probabilities of occurrence $p(x = \pm 1|1)$, $p(y = \pm 1|2)$ to Bob. Bob then chooses to perform PV measurement $\{\Pi_X(x') = \frac{1}{2}(\mathbb{1} + x' X); x' = \pm 1\}$ at time $t_3 = 2\pi/\omega$ on the assemblage $\{\rho_{x=\pm 1|1}\}$ (note that at $t_3 = 2\pi/\omega$, the

observable $X_3 = X$); he obtains the conditional probabilities for his measurement outcomes $x' = \pm 1$:

$$\begin{aligned}\mathcal{P}(x'|x) &= \text{Tr}[\rho_{x|1} \Pi_X(x')]/p(x) \\ &= \text{Tr}[E_X(x) \Pi_X(x')] \\ &= \frac{1}{2}(1 + \eta x x').\end{aligned}\quad (11)$$

Further, Bob carries out PV measurements $\{\Pi_Y(y') = \frac{1}{2}(\mathbb{1} + y' Y); y' = \pm 1\}$ at time $t_4 = 5\pi/(2\omega)$ on the assemblage $\{\rho_{y=\pm 1|1}\}$ (at $t_4 = 5\pi/(2\omega)$, the observable $X_4 = Y$) and registers the conditional probabilities for his measurement outcomes $y' = \pm 1$:

$$\begin{aligned}\mathcal{P}(y'|y) &= \text{Tr}[\rho_{y|2} \Pi_Y(y')]/p(y), \\ &= \text{Tr}[E_Y(y) \Pi_Y(y')] \\ &= \frac{1}{2}(1 + \eta y y').\end{aligned}\quad (12)$$

(In the second lines of (11), (12), we have substituted the probabilities of Alice's outcomes $p(x) = \text{Tr}[\rho E_X(x)] = 1/2$, $p(y) = \text{Tr}[\rho E_Y(y)] = 1/2$ in the qubit state $\rho = \frac{1}{2}$.)

As the expectation values of the qubit observables $(X + Y)/\sqrt{2}$ in any arbitrary qubit state is constrained to be less than 1 (the maximum eigenvalue of the observable), evidently the conditional expectation value of the observable $(X + Y)/\sqrt{2}$ (evaluated from Bob's measurement outcomes x' , y' of X , Y – which are conditioned by Alice's POVM outcomes x , y of the same observables) too is restricted i.e.,

$$\begin{aligned}\left\langle \frac{(X + Y)}{\sqrt{2}} \right\rangle_{x,y} &= \frac{1}{\sqrt{2}} (\langle X \rangle_x + \langle Y \rangle_y) \\ &= \frac{1}{\sqrt{2}} \left[\sum_{x'} \mathcal{P}(x'|x) x' + \sum_{y'} \mathcal{P}(y'|y) y' \right] \leq 1.\end{aligned}\quad (13)$$

If the assemblages $\{\rho_{x=\pm 1|1}\}$, $\{\rho_{y=\pm 1|1}\}$, obtained after Alice performs her measurements, constitute a HS structure i.e., $\rho_{x=\pm 1|1} = \sum_{\lambda} g(\lambda) p(x|1; \lambda) \rho_{\lambda}$ and $\rho_{y=\pm 1|1} = \sum_{\lambda} g(\lambda) p(y|1; \lambda) \rho_{\lambda}$, then Bob's measurements lead to a linear temporal steering inequality (obtained following the arguments outlined by Cavalcanti *et al.*²² for the derivation of linear EPR steering criteria for two spatially separated qubits):

$$\left| \sum_{x=\pm 1} p(x) x \langle X \rangle_x + \sum_{y=\pm 1} p(y) y \langle Y \rangle_y \right| \leq \sqrt{2} \quad (14)$$

Substituting the conditional probabilities (11) and (12) to evaluate the expectation values $\langle X \rangle_x$, $\langle Y \rangle_y$ and simplifying, the linear temporal inequality (14) results in the constraint

$$\eta \leq \frac{1}{\sqrt{2}} \quad (15)$$

on the unsharpness parameter – which is exactly the condition for joint measurability^{6,11,44} of the qubit observables X, Y using the POVMs $\{E_X(x) = \frac{1}{2}(\mathbb{1} + \eta x X); x = \pm 1\}$ and $\{E_Y(y) = \frac{1}{2}(\mathbb{1} + \eta y Y); y = \pm 1\}$. The temporal steering inequality (14) is violated for $\frac{1}{\sqrt{2}} < \eta \leq 1$ i.e., when Alice’s POVMs are incompatible.

4. Conclusions

We have illustrated temporal steering phenomena in a single quantum system by developing the notion of a HS structure – which is analogous to the LHS model for spatially separated systems. Falsification of HS model implies temporal steerability. Extending the arguments of recent papers^{16,17} we have established a relation between incompatibility of quantum measurements and temporal steering phenomena. Our results highlight that a set of measurements are incompatible if and only if they can be used to demonstrate temporal steering in any quantum state. The connection between measurement incompatibility and temporal steering opens up new avenues for exploring temporal steering inequalities to infer about (non) joint measurability. Further, following similar lines of investigations on non-local steering vs Bell non-locality²⁶ of spatially separated states, it would be of interest to investigate if Leggett-Garg inequalities and temporal steering inequalities carry identical inferences about measurement invasiveness⁴² or if they bring forth its different manifestations. We leave open these aspects for future investigations.

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References

1. P. Busch, “Unsharp reality and joint measurements for spin observables,” *Phys. Rev. D* **33**, 2253–2261 (1986).
2. P. Busch, P. Lahti, and P. Mittelstaedt, “The quantum theory of measurement”, Springer-Verlag, London, 1991.
3. P. Lahti, “Coexistence and joint measurability in quantum mechanics,” *Int. J. Theor. Phys.* **42**, 893–906 (2003).
4. E. Andersson, S. M. Barnett, and A. Aspect, “Joint measurements of spin, operational locality, and uncertainty,” *Phys. Rev. D* **72**, 042104 (2005).
5. W. Son, E. Andersson, S. M. Barnett, and M. S. Kim, “Joint measurements and Bell inequalities,” *Phys. Rev. A* **72**, 052116 (2005).
6. T. Heinosaari, D. Reitzner, and P. Stano, “Notes on joint measurability of quantum observables,” *Found. Phys.* **38**, 1133–1147 (2008).
7. S. Twareque Ali, C. Carmeli, T. Heinosaari, and A. Toigo, “Commutative POVMs and fuzzy observables,” *Found. Phys.* **39**, 593–612 (2009).
8. M. M. Wolf, D. Perez-Garcia, and C. Fernandez, “Measurements incompatible in quantum theory cannot be measured jointly in any other no-signaling theory,” *Phys. Rev. Lett.* **103**, 230402 (2009).
9. S. Yu, N. Liu, L. Li, and C. H. Oh, “Joint measurement of two unsharp observables of a qubit,” *Phys. Rev. A* **81**, 062116 (2010).
10. T. Heinosaari, and M. M. Wolf, “Nondisturbing quantum measurements,” *J. Math. Phys.* **51**, 092201 (2010).
11. Y. C. Liang, R. W. Spekkens, and H. M. Wiseman “Specker’s parable of the overprotective seer: A road to contextuality, nonlocality and complementarity,” *Phys. Rep.* **506**, 1–39 (2011).
12. D. Reeb, D. Reitzner, and M. M. Wolf, “Coexistence does not imply joint measurability,” *J. Phys. A: Math. Theor.* **46**, 462002 (2013).
13. M. Banik, Md. R. Gazi, S. Ghosh, and G. Kar, “Degree of complementarity determines the nonlocality in quantum mechanics,” *Phys. Rev. A* **87**, 052125 (2013).
14. R. Kunjwal and S. Ghosh, *Phys. Rev. A*. “Minimal state-dependent proof of measurement contextuality for a qubit,” *Phys. Rev. A* **89**, 042118 (2014).
15. R. Kunjwal, C. Heunen, and T. Fritz, “Quantum realization of arbitrary joint measurability structures,” *Phys. Rev. A* **89**, 052126 (2014).
16. M. T. Quintino, T. Vértesi, and N. Brunner, “Joint measurability, Einstein-Podolsky-Rosen steering, and Bell nonlocality” *Phys. Rev. Lett.* **113**, 160402 (2014).
17. R. Uola, T. Moroder, and O. Gühne, “Joint measurability of generalized measurements implies classicality” *Phys. Rev. Lett.* **113**, 160403 (2014).
18. H. S. Karthik, A. R. Usha Devi and A. K. Rajagopal, “Joint measurability, steering, and entropic uncertainty” *Phys. Rev. A* **91**, 012115 (2015).
19. J. S. Bell, “On the Einstein Podolsky Rosen paradox,” *Physics* **1**, 195–200 (1964).
20. S. Kochen and E. P. Specker, “The problem of hidden variables in quantum mechanics,” *J. Math. Mech.* **17**, 59–87 (1967).
21. E. Schrödinger, “Discussion of probability relations between separated systems,” *Proc. Camb. Phil. Soc.* **31**, 555–563 (1935).
22. For a comprehensive historical outline and for a formalization of the concept of quantum steering in the modern terminology, see E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, “Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox,” *Phys. Rev. A* **80**, 032112 (2009).
23. J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, “Proposed experiment to test local hidden-variable theories,” *Phys. Rev. Lett.* **23**, 880–884 (1969).
24. Conventionally, the notion of quantum contextuality²⁰ is demonstrated by using projective measurements in Hilbert spaces of dimension $d \geq 3$.

25. M. D. Reid, “Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification,” *Phys. Rev. A* **40**, 913–923 (1989).
26. H. M. Wiseman, S. J. Jones, and A. C. Doherty, “Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox,” *Phys. Rev. Lett.* **98**, 140402 (2007).
27. It could be that Alice randomly draws states ρ_λ with probability $g(\lambda)$ from a chosen ensemble and sends it to Bob. And when Bob asks her to perform measurement of the observable X_k , she could communicate a fake outcome x_k to have occurred with the probability $p(x_k|k) = \sum_\lambda g(\lambda) p(x_k|k, \lambda)$ (based on her knowledge of a pre-existing LHS model). In this scenario, Bob could detect that steering inequalities²² are not violated – which in turn reveals that LHS model (1) indeed holds.
28. A. J. Leggett and A. Garg, “Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks?” *Phys. Rev. Lett.* **54**, 857–860 (1985).
29. Interestingly, correlations in the measurement outcomes of time-separated observables in a single system mimic the non-local correlations in a spatially separated entangled system (see Č. Brukner, S. Taylor, S. Cheung, and V. Vedral, “Quantum entanglement in time,” *quant-ph/0402127* (2004); V. Vedral, “Using temporal entanglement to perform thermodynamical work,” *arXiv:1204.5559* (2012); J. Kofler and Č. Brukner, “Condition for macroscopic realism beyond the Leggett-Garg inequalities” *Phys. Rev. A* **87**, 052115 (2013); A. R. Usha Devi, H. S. Karthik, Sudha, and A. K. Rajagopal, “Macrorealism from entropic Leggett-Garg inequalities,” *Phys. Rev. A* **87**, 052103 (2013); C. Emary, F. Nori, and N. Lambert, “Leggett-Garg inequalities,” *Rep. Prog. Phys.* **77**, 016001 (2014).
30. A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, “Experimental violation of a Bell’s inequality in time with weak measurement” *Nat. Phys.* **6**, 442–447 (2010).
31. V. Athalye, S. S. Roy, and T. S. Mahesh, “Investigation of the Leggett-Garg inequality for precessing nuclear spins,” *Phys. Rev. Lett.* **107**, 130402 (2011).
32. G. Waldherr, P. Neumann, S. F. Huelga, F. Jelezko, and J. Wrachtrup, “Violation of a temporal Bell inequality for single spins in a diamond defect center,” *Phys. Rev. Lett.* **107**, 090401 (2011).
33. A. M. Souza, I. S. Oliveira, and R. S. Sarthour, “A scattering quantum circuit for measuring Bell’s time inequality: a nuclear magnetic resonance demonstration using maximally mixed states,” *New. J. Phys.* **13**, 053023 (2011).
34. G. C. Knee, S. Simmons, E. M. Gauger, J. J. L. Morton, H. Riemann, N. V. Abrosimov, P. Becker, Hans-Joachim Pohl, K. M. Itoh, M. L. W. Thewalt, G. A. D. Briggs and S. C. Benjamin, “Violation of a Leggett-Garg inequality with ideal non-invasive measurements,” *Nat. Comm.* **3**, 606 (2012).
35. H. Katiyar, A. Shukla, K. R. K. Rao, and T. S. Mahesh, “Violation of entropic Leggett-Garg inequality in nuclear spins,” *Phys. Rev. A* **87**, 052102 (2013).
36. J. P. Paz and G. Mahler, “Proposed test for temporal Bell inequalities” *Phys. Rev. Lett.* **71**, 3235–3239 (1993).
37. The Leggett-Garg inequality or the *temporal* Bell inequality involves linear combinations of correlations between temporally separated observables and is violated by almost every dynamically evolving quantum system (with an appropriate choice of observable measured at different intervals of time).
38. H. S. Karthik, A. R. Usha Devi, J. Prabhu Tej and A. K. Rajagopal, “Entropic uncertainty assisted by temporal memory” *arXiv:1310.5079* (2013).
39. Y. N. Chen, C. M. Li, N. Lambert, S. L. Chen, Y. Ota, G. Y. Chen, and F. Nori, “Temporal steering inequality” *Phys. Rev. A* **89**, 032112 (2014).
40. Temporal steering inequalities could be formulated by considering measurements of observables at different time instants in a single quantum system, as a natural temporal counterpart of EPR non-local steering inequalities. Recently, Chen *et. al.*³⁹ investigated the violation of the temporal analog of a quadratic EPR steering inequality (see D. H. Smith, G. Gillett, M. P. Almeida, C. Branciard, A. Fedrizzi, T. J. Weinhold, A. Lita, B. Calkins, T. Gerrits, H. M. Wiseman, S. W. Nam and A. G. White, “Conclusive quantum steering with superconducting transition-edge sensors,” *Nat. Commun.* **3**, 625 (2012)) given by $S_N = \sum_{i=1}^3 \{ \sum_{a=\pm 1} p(a) \langle B_i \rangle_{A_i=a}^2 \} \leq 1$, where $\langle B_i \rangle_{A_i=a} = \sum_b p(b|a) b$; A_i, B_i are qubit observables with binary outcomes $a, b = \pm 1$ in $N = 2, 3$ mutually unbiased basis and S_N denotes the steering parameter.
41. One could also extend the temporal steering protocol in a single quantum system, by considering measurements of any arbitrary non-commuting observables X_k (which need not be associated with dynamical observable X at time t_k). We have emphasized the *temporal* scenario here to highlight a natural counterpart of the EPR non-local steering (in spatially separated systems) for time separated observables in a single quantum system.
42. Invasiveness of measurements of an observable at different instants of time can be identified to be the temporal analog of *non-locality* of spatially separated systems³⁹.
43. Note that $\rho_{x_k|k} = \frac{1}{d} E_k(x_k) > 0$ and the set of operators $\{d\rho_{x_k|k}\}$ obey $\sum_k d\rho_{x_k|k} = \sum_k E_k(x_k) = \mathbb{1}$.
44. Necessary and sufficient conditions for the joint

measurability of a generic set of N dichotomic qubit POVMs $\{E_k(x_k), x_k = \pm 1; k = 1, 2, \dots, N\}$ have been derived in Ref.^{11,14}, by identifying the range of

the unsharpness parameter η for which the POVMs are compatible/incompatible.